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On the latest dimension theory

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by

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Although dimension theory for separable metric spaces was established about 30 years ago by Brouwer, Menger, Urysohn, Alexandroff and other mathematicians, it had been supposed to be very difficult to establish a dimension theory for non-separable cases. In about 1950, the sum-theorem, which took a leading role in the old dimension theory, was extended to normal spaces by K. Morita [1] and other mathematicians. Following that development, a satisfactory dimension theory was established for non-separable metric spaces by M. Katětov (1952) [2] and K. Morita (1954) [3]. The main part of their theory consists of extensions of theorems that were well known for separable metric spaces, but we must notice that those extensions are possible by virtue of the latest development in general topology and especially in the theory of coverings.

To begin with let us give three definitions of dimensions.

Definition 1. (Menger-Urysohn's weak inductive dimension)

i ind dim $\emptyset = -1$

ii if for any neighbourhood $U(p)$ of any point p of a topological space R, there exists an neighbourhood $V(p)$ of p satisfying $p \in V(p) \subset U(p)$,
ind dim of the boundary of $V(p) \leqq n-1$, then

$$\text{ind dim } R \leqq n.$$

Definition 2. (Strong inductive dimension)

i Ind dim $\emptyset = -1$

ii if for any closed sets F, G with $F \cap G = \emptyset$,

there exists an open set U such that $F \subset U \subset R-G$,
Ind dim of the boundary of $U \leqq n-1$, then

$$\text{Ind dim } R \leqq n.$$

Definition 3. (Lebesgue's covering dimension)

If for every finite open covering $\mathfrak{U}$ of R there exists an open covering $\mathfrak{V}$ with $\mathfrak{V} < \mathfrak{U}$, order $\mathfrak{V} \leqq n+1$, then dim $R \leqq n$.

Def.2 and Def.3 are especially important for the latest dimension theory. Actually Def.2 = Def.3 for every metric space R' was proved by Katětov and Morita, but whether Def.1 = Def.2 for every metric space or not, is as yet a difficult open problem.

From now forth we will concern ourselves only with metric spaces. The following principal theorems are extended from separable metric spaces to general metric spaces in Katětov-Morita's theory.

Sum-theorem. Let R_i, $i=1,2\dots$ be closed subsets with $\dim R_i \leq n$, then $\dim \bigcup_{i=1}^{\infty} R_i \leq n$.

Decomposition-theorem. $\dim R \leq n$ if and only if there exist $n+1$ 0-dimensional subspaces R_i, $i=1\dots n+1$ such that $R = \bigcup_{i=1}^{n+1} R_i$.

Product-theorem. $\dim R_1 \times R_2 \leq \dim R_1 + \dim R_2$.

On the other hand, the following is Menger-Urysohn's inbedding theorem whose extension to non-separable spaces was left open even in Katětov-Morita's theory.

Imbedding-theorem. A separable metric space R has dimension $\leq n$ if and only if it is homeomorphic to a subset of the set E^n_{2n+1} of points in E_{2n+1} at most n of whose coordinates are rational.

This problem was recently solved by the speaker [4] by imbedding every metric space R with $\dim R \leq n$ into a subset S_n of the generalized Hilbert space H, which was set up by C.H. Dowker.

Although most of the abovementioned theorems are just extensions of theorems that have been well known for separable metric spaces, there are also quite new types of theorems that had been unknown even for separable spaces and were established for general metric spaces quite recently.

The speaker will devote the rest of his lectures to those new developments of the theory.

First of all let us recall some of the new test theorems for n-dimensionality. K. Morita proved in 1954 [3] the following theorem.

Theorem 1. $\dim R \leq n$ if and only if there exists an open base $\mathfrak{U}$ such that $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$, $\mathfrak{U}_i$ is a locally finite system of open sets, order $\{\bar{U} - U \mid U \in \mathfrak{U}\} \leq n$. It is very interesting to find an analogy between this theorem and the metrizability theorem due to Yu. M. Smirnov and the speaker.

The following test theorem due to the speaker in 1956 [5] has also a remarkable analogy with the metrizability theorem of P. Alexandroff and P. Urysohn.

Theorem 2. $\dim R \leq n$ if and only if there exists a sequence $\mathfrak{U}_i$, $i=1,2\dots$ of open coverings such that

i $\quad \mathfrak{U}_1 > \mathfrak{U}^*_2 > \mathfrak{U}_2 > \mathfrak{U}^*_3 > \dots,$

ii $\quad \{S(p, \mathfrak{U}_i) \mid i = 1,2,\dots\}$ is a neighbourhood base of each point p of R.

iii order $\mathfrak{U}_i \leqq n+1$, $i=1,2\ldots$

W. Hurewicz and C.H. Dowker also proved in 1956 [6] a similar theorem.

Theorem 3. $\dim R \leqq n$ if and only if there exists a sequence $\mathfrak{U}_i$, $i=1,2\ldots$ such that

i for every $U \in \mathfrak{U}_{i+1}$ there exists $V \in \mathfrak{U}_i$ satisfying $\overline{U} \subset V$,

ii $\operatorname{mesh} \mathfrak{U}_i = \max.\ \text{diameter} \{\overline{U} \mid U \in \mathfrak{U}_i\} \longrightarrow 0$ as $i \to \infty$,

iii order $\mathfrak{U}_i \leqq n+1$, $i=1,2,\ldots$.

We must notice the fact that these theorems have greatly simplified the test for n-dimensionality of metric spaces, because it suffices to show just the existence of a sequence of covering to prove the n-dimensionality of a metric space by virtue of these theorems.
On the other hand, if we restrict ourselves, for example, to the definition of covering dimension, then we must show the existence of a refinement $\mathfrak{V}$ with order $\mathfrak{V} \leqq n+1$ for every open covering $\mathfrak{U}$.

To review some of the results deduced from these new test theorems, let us recall some definitions.
Definition 4. A space R is called countably dimensional with respect to the decomposition theorem, if it is expressed as a union of countably many 0-dimensional subspaces.
Definition 5. Let Ω be a set. Let $N(\Omega) = \{(x_1,x_2,\ldots) \mid x_i \in \Omega, i=1,2\ldots\}$. We define a metric $\rho(x,y)$ for two points $x=(x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$ of $N(\Omega)$ by $\rho(x,y) = \dfrac{1}{\min\{i \mid x_i \neq y_i\}}$.
Then $N(\Omega)$ makes a 0-dimensional metric space and is called a generalized Baire's 0-dimensional space.

The speaker succeeded in 1958 [7] by virtue of theorem 1 to extend a dimension theory from finite dimensional metric spaces to countably dimensional metric spaces, which had been an open problem till then even for separable metric spaces. The following is one of his principal results.
Theorem 4. A metric space R is countably dimensional if and only if there exists a subset S of $N(\Omega)$ for suitable Ω and a closed continuous mapping f of S onto R such that for each point p of R, the inverse image $f^{-1}(p)$ consists of finitely many points.

It is also an interesting problem how to characterize the dimension of a metric space by the metric which the space allows. This idea was realized, as for general metric spaces, first by J. de Groot and H. de Vries [8] who proved the following theorem in 1955.

Theorem 5. A metric space R is 0-dimensional if and only if it allows a non-Archimedean metric.

The speaker generalized this theorem by virtue of theorem 2, in 1956 [5] as follows.

Theorem 6. A metric space R has dimension $\leq n$ if and only if it allows a metric $\rho(x,y)$ such that for every $\varepsilon > 0$ and for every point p of R,

$$\rho(S_{\varepsilon/2}(p), q_i) < \varepsilon, \quad i=1\ldots n+2$$

imply $\rho(q_i, q_j) < \varepsilon$ for some i, j with $i \neq j$, where $S_{\varepsilon/2}(p)$ denotes the $\frac{\varepsilon}{2}$ neighbourhood with the center at p.

J. de Groot simplified this theorem in 1958 [9] as follows.

Theorem 7. A separable metric space R has dimension $\leq n$ if and only if it allows a metric $\rho(x,y)$ such that for every $\varepsilon > 0$ and for every point p of R,

$$\rho(p, q_i) < \varepsilon \qquad i = 1\ldots n+2$$

imply $\rho(q_i, q_j) < \varepsilon$ for some i,j with $i \neq j$.

Theorem 1 and theorem 2 or 3 are showed to be useful for proving some types of imbedding theorems, too.

Finally the speaker would like to inform of new progresses in the theory on dimension and mapping which are due to K. Nagami and J. Suzuki [10]. One of their results is the following which refines the well-known theorem of W. Hurewicz.

Theorem 8. Let f be a closed continuous mapping of a metric space R onto a metric space S such that $f^{-1}(q)$ consists of just k points for every point q of S; then dim R = dim S.